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Morphological Mesh Filtering and α -objects

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Abstract

Since image analysis techniques have come to maturity, mesh analysis has remained challenging requiring more and more efforts for elaborating an effective theoretical model. In this article, following algebraic mesh operators, we introduce algorithms that perform morphological transformations on unorganized point sets connected by their Delaunay triangulations. We show that these algorithms correspond to morphological operators like erosion, dilation or opening, acting as "shape filters" on meshes. More theoretically, a link is established between these algorithms and the formalisms of edge algebra and α -objects. Then, the mesh operators are defined in terms of complete lattices. These algorithms are applied to the problem - among others - of scene reconstruction by stereoscopy in which objects are represented by unstructured and noisy clouds of 3D points. Rapid prototyping should also benefit from these algorithms.

Key words: Shape analysis, mesh analysis, unorganized point cloud, surface-oriented representation, simplicial representation, morphological operator

1. Introduction

Scientific and industrial computing, like advanced digital manufacturing for instance, often deals with data which are a finite point set in two or three dimensional spaces. Computing what one might call the shape of such a set is very useful for visualization purpose as in the case of stereoscopic problems of reconstruction where scenes are available as a set of 3D unorganized points (Fig. 1) but also for the creation of realistic 3D models of real objects.

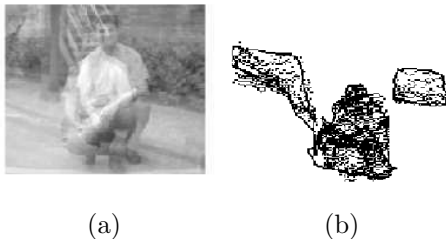


Fig. 1. (a) Superimposed stereoscopic images (b) Segmentation of a corresponding 3D point subset in three clusters and an approximation of the shape of a silhouette for each cluster.

The problem of segmenting such unorganized 3D point sets in terms of obstacles and planar navigable areas has already been addressed in (Lomenie *et al.*, 1999) by using specific fuzzy K-means based algorithms (Gath & Geva, 1989). This resulted in a point set partition. Each cluster was assigned to an object in the scene. Then the last step consisted in extracting the shape of this various objects by the means of a 3D mesh representation. But what would the formal definition of the shape of a point set be? Some works use a classical skeleton representation extracted from a Voronoï diagram (Cloppet *et al.*, 2000), but, with this question in view, 2D or 3D α -shapes first introduced by H. Edelsbrunner (Edelsbrunner & Kirkpatrick, 1983) give a formal definition of the shape of point sets as a generalized convex hull representation. These authors design shapes by sculpting Delaunay triangulations. For the moment we chose to work only on 2D point sets for the following reasons :

- we refer to one of the papers of N. Amenta's team (Amenta & Bern, 1999) that claims that the 3D extension of 2D operators is far from being as direct and robust as expected.
- in most cases, reconstructing an approximation of the 2D silhouette of an object (by projecting the 3D point set onto an approximation plane) is sufficient to get a 3D mesh of the object by back-projection.

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- mesh representations are more effective when points are distributed in the interior of the object rather than on its boundary (Amenta *et al.*, 1998). We simulate this situation by projecting the 3D boundary point sets onto specific planes.

All these issues led us to focus on 2D shape operations on unorganized point sets, which is already a challenging task. As a matter of fact, image analysis techniques on intensity regular mesh, like classical bitmap intensity images are quite mastered. Some works deal with general graph morphology operators focusing on vertices in the theoretical framework of complete lattices (Serra, 1988)(Heijmans & Ronse, 1990)(Dougherty, 1993). But the equivalent operators for geometrical irregular mesh have not been explored extensively. As mentioned above, H. Edelsbrunner was one of the first to attempt to give a formal frame to mesh manipulations : α -objects are based on Delaunay triangulations and act as thresholding operators specifically designed for meshes. In (Melkemi & Djebali, 2001b)(Melkemi & Djebali, 2001a), further investigations are carried out on this specific concept. Section 3 details the structures that will be related to the specific morphological operators designed in this paper.

2. Related work on geometrical analysis of point sets

For a decade, a new paradigm has emerged for modeling objects, such as those produced with 3D scanners : point-based surfaces. This model focuses on unorganized point sets without explicit topology, partly owing to graphics hardware considerations. Meanwhile, in the industry, for fast rendering of geometric models, explicit subdivision surface modelings based on a mesh representation have so far overtaken implicit representations such as radial basis functions. Thus, point set description of surfaces in the framework of polygonal meshes is likely to benefit from the most advanced graphics devices such as those related to GPU programming (Boubekeur *et al.*, 2006). Hence, point modeling is given more and more attention not only in the field of computational geometry but also in the field of image analysis and related topics. For the sake of illustration, we can mention PointShop3D (Zwicker *et al.*, 2002), a platform aimed at the processing of point sets comparable to usual platforms for 2D image processing such as The Gimp.

In fact, on top of visualization purposes, over the past few years, some teams have explored other aspects of mesh manipulation. The recently elaborated power crust algorithm (Amenta *et al.*, 2001)(Mederos *et al.*, 2005) improves the formal definition of the optimal parameter α related to a local density estimator inside the mesh. Thus, N. Amenta's team have developed geometrical tools for the recovery of 2D and 3D shapes of object boundary from a sufficiently dense point sample. They have pointed out the difficulties due to any 3D extension of 2D operators in computational geometry. Even though anterior α -shape structures are of-

ten unsatisfactory for reconstructing surfaces, they tend to work well for sample points which are evenly distributed in the interior of an object (Amenta *et al.*, 1998). This is the reason why this study relies on the initial concept of α -shape. But to handle and analyze this graph structure, we need an appropriate, flexible algorithmic data structure.

L. Guibas and J. Stolfi (Guibas & Stolfi, 1985) have already proposed an algebraic structure to create and manipulate any kind of geometrical mesh. Quadedge data structure is a particularly elegant data structure for polyhedra. It can be used to represent a manifold where edges play the leading role since they store the complete topological information. Vertices hold most of the geometric information. The quadedge data structure was given this name because the duality is built in at a low level by storing together quadruples of directed edges. Thus, the dual of a polyhedra is very easily computed by replacing vertices with faces and faces with vertices. In this data structure, the dual Delaunay and Voronoï graphs are structurally stored in the same computation step; besides, neighborhood relationships and accesses are explicitly stored in this structure. This is exactly what we need to perform morphological analysis on point sets.

Actually, image processing frequently uses mathematical morphology for basic topological and geometric operations but, as noted by (Paternell *et al.*, 2003)(Pottmann *et al.*, 2004) few contributions only extend morphology to curved manifolds and to meshes and cell decompositions on curved manifolds (Roerdink, 1990)(Roerdink, 1994)(Rossl *et al.*, 2000). In (Vincent, 1989)(Heijmans *et al.*, 1992), Vincent *et al.* explore graph morphological operators. But, in their theory, labeled vertices are at the center of the process. Their point of view - structural morphology focusing on the graph structure - is slightly different from ours where triangles play the leading role. Our study deals more with the original notion of geometric morphological operators for meshes. We perform morphological transformations both on the edge and the triangular face structures rather than on the vertex structure.

This paper presents new mesh morphological operators derived from preliminary works designed for unorganized point sets (Lomenie *et al.*, 2000; Lomenie, 2001) and it is based on the algebraic quadedge structure. Therefore, this paper deals with surface-like point clouds and triangular meshes. Section 3 details the concept of α -objects that we use as a thresholded mesh representation of the point set. Then, in section 4, we expose the algorithms designed for erosion, dilation, opening and closing that act on such meshes. We discuss as well the equivalence of the designed operators - geometric morphology - with those of classic mathematical morphology in terms of complete lattices. Section 5 presents some results and possible applications with an essential but limited discussion on implementation issues. Finally, section 6 concludes with some theoretical and technical perspectives that we aim at developing in the future.

3. α -Objects

We begin by explaining what an α -shape and its derived objects are, and how they are obtained (Edelsbrunner & Mücke, 1994). Our main purpose is to relate the morphological operators designed in section 4 to this specific data structure partly because both rely on set relations as \subset , \cap .

First introduced by Edelsbrunner (Edelsbrunner & Kirkpatrick, 1983), the notion of α -shape gives a formal definition of what the shape of a point cloud could be. More precisely, it defines a discrete family of shapes whose detail level is regulated by the parameter α that controls the maximum "curvature" allowed in shape description.

For a detailed description of the topological and geometrical concepts, we refer the reader to (Edelsbrunner & Mücke, 1994) but we recall here some important formal definitions limited to 2D structures.

3.1. Topological context

α -shapes are a generalization of the convex hull of a point set. Let S be a point set in \mathbb{R}^2 . For α in $[0, \infty)$, the α -shape S_α of S is a polytope which, as α decreases, gradually develops cavities. Notice that :

$$S_\infty = \text{conv}(S), \text{ where conv stands for the convex hull and } S_0 = S \quad (1)$$

In the following, ∂ stands for the boundary of a structure.

α -ball. For $0 < \alpha < \infty$, let an α -ball B_α be an open ball of \mathbb{R}^2 with radius α . B_0 is a point and B_∞ is an open half-space.

B_α is empty if $B_\alpha \cap S = \emptyset$. Such an α -ball is denoted B_α^\emptyset .

k -simplex. Let us define k -simplices $\sigma_T = \text{conv}(T)$, $T \subseteq S$ and $|T| = k + 1$ for $0 \leq k \leq 2$. Thus, a vertex is a 0-simplex, an edge is a 1-simplex, and a triangle is a 2-simplex. The *boundary* of the k -simplex σ_T consists of all $k + 1$ sub-simplices of dimension $k - 1$. The boundary of a triangle is three edges and the boundary of an edge is two vertices. Note that sometimes we will note σ_T^k to indicate explicitly the dimension of the simplex.

k -faces and S_α . For all k -simplex σ_T , $0 \leq k \leq 1$,

$$\sigma_T \text{ is } \alpha\text{-exposed} \iff \exists B_\alpha^\emptyset / T = \partial B_\alpha^\emptyset \cap S \quad (2)$$

Thus, a fixed α defines sets $F_{k,\alpha}$ of α -exposed k -simplices for $0 \leq k \leq 1$. Thus, the α -shape S_α of S is the polytope whose boundary consists of the edges in $F_{1,\alpha}$ and the vertices in $F_{0,\alpha}$:

$$\partial S_\alpha = \bigcup_{0 \leq k \leq 1} F_{k,\alpha} \quad (3)$$

The k -simplices in $F_{k,\alpha}$ are also called the k -faces of S_α .

Simplicial complex. A simplicial complex is a collection C of closed k -simplices, for $0 \leq k \leq 2$, that satisfies the following properties :

– If $\sigma_T \in C$ then $\sigma_{T'} \in C$ for every $T' \subseteq T$.

– If $\sigma_T, \sigma_{T'} \in C$, then either $\sigma_T \cap \sigma_{T'} = \emptyset$ or $\sigma_T \cap \sigma_{T'} = \sigma_{T \cap T'} = \text{conv}(T \cap T')$

A subset $C' \subseteq C$ is a subcomplex of C if it is also a simplicial complex.

α -hull. We can define related geometrical structures such as the α -convex hull H_α of S :

$$H_\alpha(S) = \left\{ \bigcup B_\alpha^\emptyset \right\}^C \quad (4)$$

Then we have the following property :

$$H_\infty(S) = \text{conv}(S) = S_\infty(S) \quad (5)$$

Note at this point that mathematical morphology is also well known for its set relations : " $B \subset S$ ", " $B \cap S \neq \emptyset$ ", where S is the set to analyze and B is the structuring element whose shape depends on analysis needs. As a matter of fact, these relations are the basis of elementary morphological operators such as erosion and dilation.

3.2. Graph context

Parallely, a finite point set S in \mathbb{R}^2 defines a specific triangulation known as the Delaunay triangulation of S decomposing the convex hull $H_\infty(S)$ of S into triangles.

Delaunay triangulation. For $0 \leq k \leq 2$, let F_k be the set of k -simplices σ_T for which there are empty balls B_α^\emptyset with $\partial B_\alpha \cap S = T$. Notice that $F_0 = S$. Then, the Delaunay triangulation $Del(S)$ of S is the simplicial complex defined by the triangles in F_2 , the edges in F_1 and the vertices in F_0 .

Del and S_α . By definition, for each k -simplex σ_T in Del , there exists values of α so that σ_T is α -exposed. Conversely, every face of S_α is a simplex of Del . This implies the relationship between the Delaunay triangulation and the boundary of S_α :

$$\begin{aligned} \text{For } 0 \leq k \leq 1, F_k &= \bigcup_{0 \leq \alpha \leq \infty} F_{k,\alpha} \\ Del(S) &= \bigcup_{0 \leq k \leq 1} F_k = \bigcup_{0 \leq \alpha \leq \infty} \partial S_\alpha \end{aligned} \quad (6)$$

α -complex. Each k -simplex σ_T of Del defines an open ball B_T bounded by the smallest sphere ∂B_T that contains all points of T . Let ρ_T be the radius of B_T , ∂B_T is the smallest circumsphere of T and ρ_T is the radius of σ_T . Then, for $0 \leq k \leq 2$ and $0 \leq \alpha \leq \infty$,

$$\begin{aligned} G_{k,\alpha} &= \{ \sigma_T \in Del / B_T \text{ empty and } \rho_T < \alpha \} \\ \text{and } \forall \alpha, G_{0,\alpha} &= S \end{aligned} \quad (7)$$

Then, we define the α -complex C_α of S as the simplicial complex defined this way :

$$\begin{aligned} C_\alpha(S) &= \{ \sigma_T \in Del / \sigma_T \in \bigcup_{0 \leq k \leq 2} G_{k,\alpha} \text{ or} \\ &\quad \sigma_T \in \partial \sigma_T^{k+1} \text{ with } \sigma_T^{k+1} \in C_\alpha \} \end{aligned} \quad (8)$$

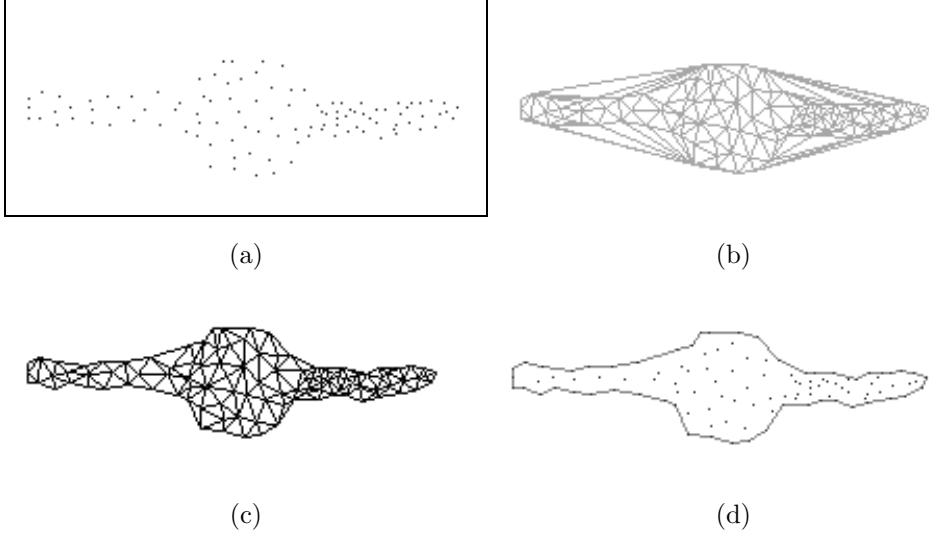


Fig. 2. (a) 2D point set S (b) Delaunay triangulation $Del(S)$ or ∞ -complex(S) (c) α_{opt} -complex (as a simplicial complex) $C_{\alpha_{opt}}(S)$ triangulated by $Del_{\alpha_{opt}}(S)$ (d) α_{opt} -shape as a polytope $S_{\alpha_{opt}}(S)$

By definition, C_{α_1} is a subcomplex of C_{α_2} if $\alpha_1 \leq \alpha_2$. Thus, the underlying space of C_α , denoted by $|C_\alpha|$, is defined by $|C_\alpha| = \bigcup_{\sigma_T \in C_\alpha} \sigma_T$, and thus :

$$\forall 0 \leq \alpha \leq \infty, S_\alpha = |C_\alpha| \quad (9)$$

This alternative definition of α -shapes makes the relationship between the k -simplices of Del and those of C_α more explicit. For example, for any $\sigma_T \in Del$,

$$\sigma_T \in C_\alpha \equiv \alpha \in (\rho_T, \infty] \text{ and } B_T \text{ is empty} \quad (10)$$

or

$$C_\alpha(S) \subseteq C_\infty(S) = Del(S) \quad (11)$$

At this point, we do not take into account topological issues such as isolated edges or points. We limit ourselves to what we call α -Delaunay triangulation defined this way :

$$Del_\alpha(S) = \{ \sigma_T \in Del / \sigma_T \in G_{2,\alpha} \text{ or } \sigma_T \in \partial \sigma_T^{k+1} \text{ with } \sigma_T^{k+1} \in C_\alpha \} \quad (12)$$

that is something like a proper C_α , so that :

$$\forall 0 \leq \alpha \leq \infty, S_\alpha = |C_\alpha| = |Del_\alpha| \quad (13)$$

Fig. 2 summarizes some of these topological and geometrical structures for a synthesized 2D point set.

3.3. Algorithmic context

Hence, we consider that the α -complex of a point set S can be viewed as a triangulation of the interior of the corresponding α -shape and both α -structures can be defined as subgraphs of the Delaunay triangulation Del of S . Intuitively, once the Delaunay triangulation is obtained (de Berg, 1997), the α -complex structure acts as a spherical eraser deleting triangles of Del able to receive an open ball B_α , of radius α , not containing any points of S .

Then, as explained in detail in (Edelsbrunner & Mücke, 1994), for each simplex $\sigma_T \in Del$, there is a single interval so that σ_T is a face of the α -shape S_α , i.e. if, and only if, α is contained in this interval. Let $up(\sigma_T)$ be the set of all faces incident to σ_T whose dimension is one higher than that of σ_T , that is :

$$up(\sigma_T) = \{ \sigma_{T'} \in Del \mid T \subset T' \text{ and } |T'| = |T| + 1 \}$$

Then, for each σ_T , two values λ_T and μ_T are derived :

$$\text{if } |T| = 3, \lambda_T = \mu_T = \rho_T \\ \text{else } \begin{cases} \lambda_T = \min \{ \lambda_{T'} \mid \sigma_{T'} \in up(\sigma_T) \} \\ \text{and} \\ \mu_T = \max \{ \mu_{T'} \mid \sigma_{T'} \in up(\sigma_T) \} \end{cases}$$

Let us note the similarity of these equations with rank filtering equations from image processing morphological operators. We will develop this analogy throughout this work and in particular in the next section.

Last a simplex is said to be :

$$\begin{cases} \text{Interior} & \text{if } \sigma_T \notin \partial S_\alpha \\ \text{Regular} & \text{if } \sigma_T \in \partial S_\alpha \text{ and bounds some} \\ & \text{higher dimensional simplex in } C_\alpha \\ \text{Singular} & \text{if } \sigma_T \notin \partial S_\alpha \text{ and does not bound} \\ & \text{any higher dimensional simplex in } C_\alpha \end{cases}$$

Then, Table 1 classifies the simplices of Del in order to compute α -objects in the framework of (Edelsbrunner & Mücke, 1994).

Note that by convention each edge belonging to $\partial conv(S)$ is the edge of a triangle of infinite radius with a point at infinity. Thus, theoretically and based on the classification

Table 1
Obtaining α -objects

σ_T is...	Singular	Regular	Interior
Triangle			$\alpha \in [\rho_T, \infty[$
Edge, $\notin \partial conv(S)$	$\alpha \in [\rho_T, \lambda_T[$	$\alpha \in [\lambda_T, \mu_T[$	$\alpha \in [\mu_T, \infty[$
Edge, $\in \partial conv(S)$	$\alpha \in [\rho_T, \lambda_T[$	$\alpha \in [\lambda_T, \infty[$	
Vertex, $\notin \partial conv(S)$	$\alpha \in [0, \lambda_T[$	$\alpha \in [\lambda_T, \mu_T[$	$\alpha \in [\mu_T, \infty[$
Vertex, $\in \partial conv(S)$	$\alpha \in [0, \lambda_T[$	$\alpha \in [\lambda_T, \infty[$	

established in Table 1, the α -complex C_α consists of all interior, regular, singular simplices for a given α value. Besides the interior of the α -shape S_α is triangulated only by the interior triangles. Last, the boundary of S_α is formed by the set of regular edges and their vertices :

$$\begin{aligned} C_\alpha &= \{\text{Singular } \sigma_T\} \cup \{\text{Regular } \sigma_T\} \cup \{\text{Interior } \sigma_T\} \\ Del_\alpha &= \{\text{Interior } \sigma_T\} \\ \partial S_\alpha &= \{\text{Regular } \sigma_T\} \end{aligned} \quad (14)$$

At this point, note that some topological problems due to singular simplices may not be taken into account here since we chose to emphasize geometric aspects. It appeared that simplification did not play a key role in the following applications. Thus, the boundary of the α -shape is formed by the set of regular edges and their vertices and, in this preliminary study, we do not take into account singular simplices.

The main purpose of this work is to propose new structures based on these α -structures leading to morphological analysis and transformation tools specifically designed for meshes. This will be the main subject of the next section. But while the original work on α -objects used the radius ρ_T of the circumscribed sphere as the unique measure to characterize the shape of the 2D point set, we can set the measure ρ_T in Table 1 to different values and inherit all the previous α -structures for this specific measure. For example, in what follows, ρ_T will also describe the size of the triangles rather than their shapes by setting $\rho_T = \max(AB, BC, AC)$ where A,B,C are the vertices of the triangle T. This idea will be illustrated in section 5.

4. Mesh morphological operators based on α -objects

In this paper, we mainly aim to develop triangle mesh based algorithms. Such algorithms have often focused on the global topological properties of the underlying manifold such as surface components and a complete description of boundaries when present. As a matter of fact, a fundamental result from algebraic topology states that manifold triangulations are homeomorphic to surfaces. This implies that the topological characteristics of surfaces can be determined from triangulations. Such characteristics can be defined either in terms of point sets or triangulations (Henle, 1979). For example, a surface is *compact* if its triangula-

tion contains a finite number of triangles. A *component* of a surface is a subset such that all simplices in its triangulation are reachable from any other by a continuous walk that crosses edges. A surface is *connected* if it contains exactly one component. A surface that is compact and connected is *closed* etc. In this section, an original extension of these topological characteristics to morphological characteristics computed from triangular meshes is proposed.

Once an optimal α -complex of S is obtained, for instance the one of minimum volume or regular boundary and containing all the points of S , our purpose is to define morphological operators in order to filter its shape. Experimentally, $\alpha_{opt} = 2 * \text{median}_{\sigma_T \in Del}(\rho_T)$ gives this optimal α -complex for sufficiently locally dense point sets. Such was the case for our stereoscopic data. See (Amenta *et al.*, 2001)(Mederos *et al.*, 2005) for more sophisticated methods. Hereafter, as explained in the previous section, α -complex is topologically equivalent to a triangulation of the α -shape, that is to the sub-triangulation Del_α obtained from Del .

We chose to present these new operators from two point of views :

- as a spectrum of α -objects which can be computed off-line during the computation of the Delaunay triangulation like the spectrum of α -hulls initially proposed in (Edelsbrunner & Mücke, 1994) (see section 4.2);
- as a specific case within the complete lattice framework to propose on-the-fly operators to filter 2D point sets as is done with 2D images in any image processing toolbox (see again (Zwicker *et al.*, 2002)) (see section 4.5).

4.1. Definition

In the α -object framework, we need to affect to each triangle (and in the following to each edge) a series of values e_T^k in addition to the measure ρ_T , defined by :

$$\begin{aligned} \forall k \in N, e_T^k &= \max\{e_{T'}^{k-1} | T' \in \text{neighbor}(T)\} \\ \forall k \in N, d_T^k &= \min\{d_{T'}^{k-1} | T' \in \text{neighbor}(T)\} \\ \text{and } e_T^0 &= d_T^0 = \rho_T \end{aligned} \quad (15)$$

where $\text{neighbor}(T)$ (somehow the equivalent of the definition of $up(\sigma_T)$ in the framework of α -objects) is the set of all triangles T' of Del sharing at least one vertex with the triangle T , that is :

$$\begin{aligned} \text{neighbor}(T) &= \{T' \in Del | T' \cap T \neq \emptyset \\ &\text{and } |T'| = |T| = 3\} \end{aligned} \quad (16)$$

Doing that, we define a neighborhood system on Del similar to the 8-connexity system defined on regular pavages for images.

4.2. Spectrum of new α -objects : α^k -eroded, -dilated etc.

The α^k -eroded of S is defined as a subgraph of Del obtained by propagating e_T^k values to neighbor triangles.

Thus, the spectrum of the α -eroded of order k of any point set is defined as the reunion of all the triangles of C_α whose e_T^k value is inferior to α , that is :

$$\alpha^k - \text{eroded}(S) = \{T' \in Del | e_{T'}^k < \alpha \text{ and } |T'| = 3\} \quad (17)$$

As of now, α - transformation of order 1 or α^1 - transformation will be called α -transformation. Note that α^k -eroded of Del is the same as the α -eroded of the α^{k-1} -eroded, replacing e_T^0 by e_T^{k-1} . Thus, performing α^k -erosion is the same as performing k successive α -erosion propagating e_T^k values.

The α^k -dilated of S is defined as a subgraph of Del obtained by propagating d_T^k values to neighbor triangles. Thus, the spectrum of the α -dilated of order k of any point set is defined as the reunion of all the triangles of C_α whose d_T^k value is inferior to α , that is :

$$\alpha^k - \text{dilated}(S) = \{T' \in Del | d_{T'}^k < \alpha \text{ and } |T'| = 3\} \quad (18)$$

Note that by convention the dilation of Del gives the whole 2D space. Furthermore, Fig. 3 illustrates the duality of the α -erosion and α -dilation operators. The black area in Fig. 3 (b) corresponds to the α -dilated of the complementary of the α -complex and is defined by :

$$(\alpha^k - \text{dilated}(S))^C = \{T' \in Del | e_{T'}^k > \alpha \text{ and } |T'| = 3\} \quad (19)$$

And hence, the complementary of this $\alpha^k - \text{dilated}^C$ corresponds to the black area in Fig. 3(a) that is to the α -eroded of the α -complex. In fact, the path to duality consists in inverting the operators \inf and \sup in the previous definitions.

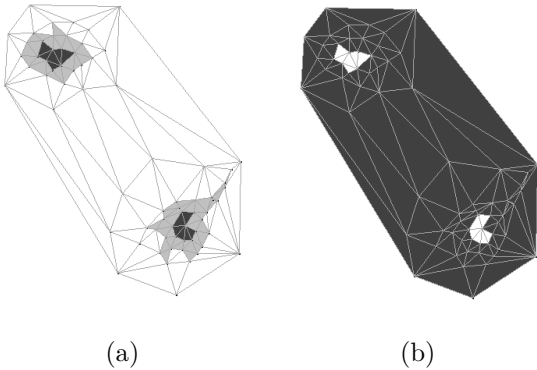


Fig. 3. Duality of the transformations : (a) in gray, the α -complex and in black, the α -eroded complex (b) in black, the α -dilated of the complementary α -complex

α -opening of order k (α^k -opening) is defined as the dilation of the α -eroded of order k of C_α . Note that in this framework performing α^k -opening is not the same as performing k successive α -opening propagating e_T^k values. It is performing k successive α -erosion followed by a dilation

in order to filter the contour of the obtained shape. In this specific case, the dilation of the α -eroded of order k of C_α can be obtained by using the functional definition of mathematical dilation. Thus, to each triangle σ_T of C_α are assigned values o_T^k defined by :

$$o_T^k = \min\{e_{T'}^k | T' \in \text{neigh}(T)\} \\ = \min_{T' \in \text{neigh}(T)} \{ \max_{T'' \in \text{neigh}(T')} \{e_{T''}^{k-1}\} \} \quad (20)$$

Again, we come across the same kind of expressions as for opening operators of classical morphology. Thus, the dilation of the α -eroded of order k , named $\alpha^k - \text{opened}$, is defined as the reunion of all the triangles of C_α whose o_T^k value is inferior to α , that is :

$$\alpha^k - \text{opened} = \{T' \in Del | o_{T'}^k < \alpha \text{ and } |T'| = 3\} \quad (21)$$

In this part, we have presented a spectrum of new α -objects and we will give an interpretation of these new structures in the framework of complete lattices in section 4.5.

4.3. Duality edge/face

The proposed framework can filter the shape of the α -complex (region mode) as much as the shape of its boundary (contour mode). In the latter case and referring to Table 1, the propagation of e_T^k and o_T^k values over the faces is then replaced by the propagation of λ_e^k , μ_e^k , λ_o^k and μ_o^k values associated with the edges defined by :

$$\lambda_e^k = \min\{e_T^k, e_{T'}^k\} \text{ and } \mu_e^k = \max\{e_T^k, e_{T'}^k\} \\ \lambda_o^k = \min\{o_T^k, o_{T'}^k\} \text{ and } \mu_o^k = \max\{o_T^k, o_{T'}^k\} \quad (22)$$

where T and T' are the two adjacent triangles of the edge E . Table 2 summarizes in the same framework as Table 1 the different inequalities allowing the computation of various filtered α -shapes in both the contour mode (∂S_α) and the region mode (Del_α) (see Fig. 7 and Fig. 8) :

Table 2
Obtaining filtered α -objects

Morph. operator	Region Mode (acting on triangles)	Contour Mode (acting on edges)
$\alpha^k - \text{eroded}$	$\alpha \in [e^k, \infty[$	$\alpha \in [\lambda_e^k, \mu_e^k]$
$\alpha^k - \text{opened}$	$\alpha \in [o^k, \infty[$	$\alpha \in [\lambda_o^k, \mu_o^k]$

4.4. Mathematical, graph and mesh morphology

The presented results show that the designed operators for Delaunay triangulations perform the same kind of transformations as classical mathematical morphology and that they seem to share the same properties. As pointed out in (Vincent, 1989) for graph morphology, the mesh morphology developed here resembles classical morphology but the

local neighborhood structure may differ at different vertices. An image can be considered as a planar graph when the regular support grid is restricted to 4-connectivity or hexagonal grids for instance whereas the concept of structuring graph (Vincent, 1989) defines an evolutive neighborhood for every vertex in a graph. Yet, this framework has not led to a lot of applications because of its computational complexity. Indeed, the manipulation of such structural structuring graphs is related to complex graph isomorphism issues and results are usually illustrated with a simple structuring graph as will be illustrated later in this section (see Fig. 5(b)). Our aim is to define a more generic, tractable framework for shape analysis of mesh inspired by these seminal insights.

In this part, we propose to illustrate the parallel between image and mesh morphology. Let us represent a gray level image I by its functional $f(x, y)$ defined on a pavage of the discretized image. This pavage is a regular neighborhood system where the notions of 4-connectivity and 8-connectivity are directly inherited as illustrated in Fig. 4(a) and Fig. 5(a). In a parallel way, let us consider a 2D unorganized point set M as illustrated in Fig. 4. First, the Delaunay triangulation of this point set defines a neighborhood system. Second, the inverse of the radiuses of the circumscribed circle to the triangles of the triangulation M will play the role of pixel intensity in the gray level image I . Therefore, in the case of image I , the sites are the pixels associated to their gray level distributed over a regular grid, and in the case of the mesh M , the sites are the triangles associated with their ρ_T values over an irregular grid.

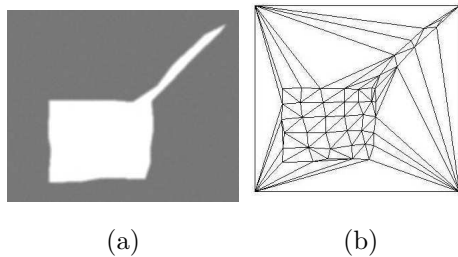


Fig. 4. Different representations of the same object : (a) gray intensity image I and (b) geometric mesh M

The first step of any radiometric image processing system could be the binarization of the input image. Fig. 6 describes this step in both intensity and geometric cases for a threshold of twice the median value of site levels. The binarization of M comes down to extracting its α -shape Del_α with α corresponding to the chosen threshold. More sophisticated works deal with the problem of extracting the external shape of dot patterns (Chaudhury *et al.*, 1997)(Amenta *et al.*, 1998)(Amenta & Bern, 1999)(Amenta *et al.*, 2001). They are based on the concept of the *structuring radius* that could be part of our prospective research. But, in this paper, only new morphological operators are considered in detail.

Once these binarized images have been obtained, one can apply the standard binary morphological operators to the

binary shapes. The shape of the structuring element used for I is drawn in Fig. 5(a). Referring to (Vincent, 1989), we should eventually think about α -erosion as an erosion with the structuring graph of Fig. 5(b). But in (Vincent, 1989) the author is much more concerned with the construction of the neighborhood function in the set of vertices of a graph, whereas in the context of a tractable application of this theory, we focus here on the simplest neighborhood function which is centered and isotropic. The results of an erosion performed on I with a structuring element of size 20 and of an α -erosion on M are described in Fig. 6(b). In the same spirit, we illustrate the results of openings in Fig. 6(c).

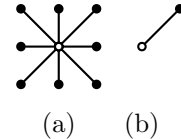


Fig. 5. Structuring element (a) and structuring graph (b)

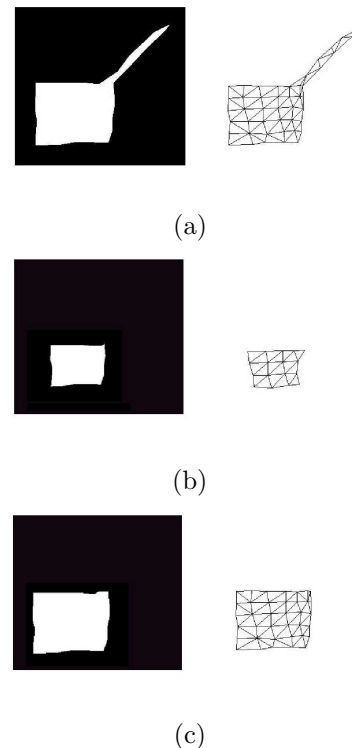


Fig. 6. Results of (a) binarization, (b) erosion and (c) opening, for both representations

4.5. Definition in the framework of complete lattices

In fact, mathematical morphology can be defined in terms of complete lattices. For these new operators to be useful for point set processing, we need to define the corresponding lattice structure. In this preliminary study, to each point set S is associated :

- its Delaunay triangulation $Del(S)$ defining the topology of the working topological subspace (playing the role of the regular pavage associated to an image);
- the set $\wp(Del)$ of all the corresponding sub-triangulations D_i of Del .

From now, we can define two complete lattice structures for a point set:

- the first one, within the set theory frame, called $\mathcal{L}_1 = (\wp(Del), \subseteq)$ where $D_1 \subseteq D_2$ denotes the relation : $\forall T \in Del, T \in D_1 \rightarrow T \in D_2$;
- the second one, within the functional theory frame, called $\mathcal{L}_2 = (\mathcal{M}(Del), \leq)$, where $\mathcal{M}(Del)$ is the set of meshes on Del , i.e., the set of mappings from the triangles T in Del to ρ_T values, and where the partial ordering \leq is defined by :

$$\forall M_1 \text{ and } M_2 \in \mathcal{M}(Del),$$

$$M_1 \leq M_2 \iff \forall T \in Del, \rho_T^1 \geq \rho_T^2 \quad (23)$$

From now, the notation T stands for any triangle in $Del(S)$ that is any simplex σ_T with $|T| = 3$ in $Del(S)$.

Let us remind that a partial ordering induces infimum and supremum operators, and that in the case of \mathcal{L}_1 and \mathcal{L}_2 the infimum operators are given by the following statements.

$$\forall D_1 \text{ and } D_2 \in \wp(Del) :$$

$$D_1 \subseteq D_2 \equiv \forall T, T \in D_1 \rightarrow T \in D_2$$

$$D_1 \cap D_2 = \{T \in Del / T \in D_2 \text{ and } T \in D_1\}$$

$$D_1 \cup D_2 = \{T \in Del / T \in D_2 \text{ or } T \in D_1\} \quad (24)$$

Thus,

$$\forall D_1, D_2 \in \mathcal{L}_1, \inf(D_1, D_2) = D_1 \cap D_2 \quad (25)$$

As for \mathcal{L}_2 , based on Eq. 23,

$$\forall M_1, M_2 \in \mathcal{L}_2, \inf(M_1, M_2) = \{T \in Del, \max(\rho_T^1, \rho_T^2)\} \quad (26)$$

Similarly, the supremum operators are given by :

$$\forall D_1, D_2 \in \mathcal{L}_1, \sup(D_1, D_2) = D_1 \cup D_2 \quad (27)$$

$$\forall M_1, M_2 \in \mathcal{L}_2, \sup(M_1, M_2) = \{T \in Del, \min(\rho_T^1, \rho_T^2)\}$$

Let us also define the binarization operator α -bin as follows :

$$\forall M \in \mathcal{M}(Del(S)),$$

$$\alpha - bin(M) = \{T \in Del \mid \rho_T < \alpha\} \quad (28)$$

and thus,

$$\alpha - bin(M) = \alpha - complex(S) \quad (29)$$

From now, we can define two operators $e(M)$ and $d(M)$ on this complete lattice \mathcal{L}_2 by :

$$\forall M \in \mathcal{M}(Del), e(M) = \{T \in Del, e_T\}$$

$$\text{and } d(M) = \{T \in Del, d_T\} \quad (30)$$

with e_T and d_T defined in Eq. 15. We can derive the following property :

$$e(\inf(M_1, M_2)) = e(\{T \in Del, \rho_T^{max} = \max(\rho_T^1, \rho_T^2)\})$$

$$= \{T \in Del, \max\{\rho_{T'}^{max} \mid T' \in neighbor(T)\}\}$$

$$= \{T \in Del, \max\{\max(\rho_{T'}^1, \rho_{T'}^2) \mid T' \in neighbor(T)\}\}$$

$$= \{T \in Del, \max(\max\{\rho_{T'}^1 \mid T' \in neighbor(T)\}, \max\{\rho_{T'}^2 \mid T' \in neighbor(T)\})\}$$

$$e(\inf(M_1, M_2)) = \inf(e(M_1), e(M_2)) \quad (31)$$

Thus, the operator $e(M)$ on \mathcal{L}_2 is distributive with respect to the *infimum*. By duality, the operator $d(M)$ is distributive with respect to the *supremum*. Hence, $e(M)$ and $d(M)$ are respectively erosion and dilation morphological operators \mathcal{L}_2 . From this point on, $e^{\mathcal{L}_1}$ and $e^{\mathcal{L}_2}$ denote the erosions defined on \mathcal{L}_1 and \mathcal{L}_2 , respectively. Last, we can now define more formally the erosion and dilation operators on the sub-triangulation complete lattice \mathcal{L}_1 . Let D be a sub-triangularization of Del . Let D be seen as a subset of the topological space Del . Then, T is an interior triangle of D (that is $T \in int(D)$) if there exists a neighborhood of T which is contained in D . Then:

$$e(D) = \{T \in Del \mid T \in int(D)\}$$

$$d(D) = \{T \in Del \mid T \notin int(D)^C\} \quad (32)$$

Again, we can derive the following property :

$$e(\inf(D_1, D_2)) = e(\{T \in Del \mid T \in D_1 \text{ and } T \in D_2\})$$

$$= \{T \in Del \mid T \in D_1 \text{ and } T \in D_2 \text{ and } T \in int(D_1 \cap D_2)\}$$

$$= \{T \in Del \mid T \in D_1 \text{ and } T \in D_2$$

$$\text{and } T \in int(D_1) \text{ and } T \in int(D_2)\}$$

$$= \{T \in Del \mid T \in D_1 \text{ and } T \in int(D_1)\} \cap$$

$$\{T \in Del \mid T \in D_2 \text{ and } T \in int(D_2)\}$$

$$e(\inf(M_1, M_2)) = \inf(e(D_1), e(D_2)) \quad (33)$$

Thus, the operator $e(D)$ on \mathcal{L}_1 is distributive with respect to the *infimum*. By duality, the operator $d(D)$ is distributive with respect to the *supremum*. Hence, $e(D)$ and $d(D)$ are respectively erosion and dilation morphological operators on \mathcal{L}_1

Last, we can state the following:

$$\forall M \in \mathcal{M}(Del(S)),$$

$$\alpha - bin(e^{\mathcal{L}_2}(M)) = e^{\mathcal{L}_1}(\alpha - bin(M)) = \alpha - eroded(S)$$

$$\alpha - bin(d^{\mathcal{L}_2}(M)) = d^{\mathcal{L}_1}(\alpha - bin(M)) = \alpha - dilated(S) \quad (34)$$

that is, computing the eroded and dilated sub-triangulations as defined in section 4.2 is equivalent to first computing the α -complex and performing the erosion and dilation directly in that geometrical structure.

As a matter of fact,

$$\alpha - bin(e^{\mathcal{L}^2}(M)) = \{T \in Del \mid e_T < \alpha\} \quad (35)$$

and

$$e^{\mathcal{L}^1}(\alpha - bin(M)) = \{T \in Del \mid \rho_T < \alpha \text{ and } T \in int(C_\alpha)\} \quad (36)$$

Besides, as a neighborhood of a triangle T in Del is a triangle set containing an open set which in turn contains the triangle T , $neighbor(T)$ is the smallest neighborhood of T . Then,

$$\begin{aligned} e_T < \alpha &\equiv \max\{\rho_{T'} \mid T' \in neighbor(T)\} \leq \alpha \\ &\equiv \rho_T < \alpha \text{ and } \exists V, \text{ a neighborhood of } T \mid V \subset C_\alpha \end{aligned} \quad (37)$$

Hence, with these lattice structures, we inherit all the properties of classical morphology and particularly for the α -opening and α -closing operators.

The extension of this framework to point sets defined on two different topological subspaces, that is, on two different Delaunay triangulations would be interesting and will be the purpose of further theoretical investigations. With this aim in view, we will need to define the union of two point sets which, so far, has been a challenging issue but constitutes the next step for the design of a general framework for morphological analysis of point sets in the lattice theory. Indeed, in this seminal study we focus on point sets defined on the same discrete topological space, that is, the same Delaunay triangulation.

5. Experimental results, implementation and applications

In this section, the resulting structures are illustrated both for analysis and synthesis purposes and notes on implementation are given.

5.1. Analysis and synthesis tools

Below are some preliminary illustrations of the way the proposed operators can filter the shape of a point set S described either as an α -complex (see Fig. 8) or as its boundary (see Fig. 7).

These results point out an interesting quality of the proposed modeling : it processes both edges and triangular faces in the same formalism.

In (Lomenie, 2004), we present some heuristics inspired by the mesh filtering ideas applied not only to the clustering and analysis of 3D stereoscopic unorganized point sets but also to the reconstruction of the 3D surface of the different obstacles detected in the scene.

In point of fact, in 3D reconstruction applications, it is useful to reconstruct the 3D shape of an object from the 2D projection of one of its silhouettes or of a set of patches along the surface (Boubekeur *et al.*, 2006)(Gopi *et al.*, 2000). The

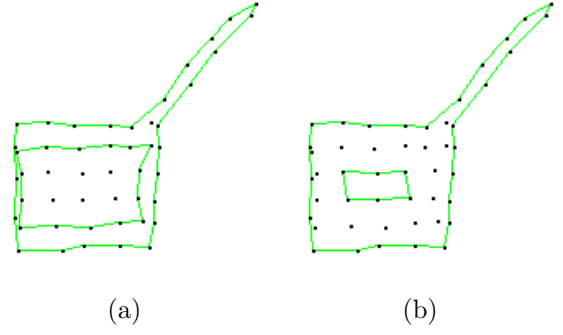


Fig. 7. Successive erosions of order 1 (a) and order 2 (b) of the same α -shape in contour mode

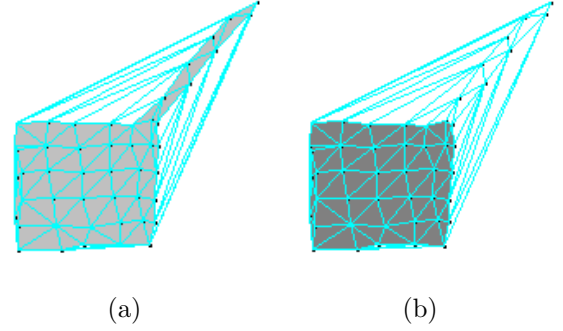


Fig. 8. (b) Opening of an (a) α -complex

morphological mesh operators designed here can be effectively applied to the filtering of this shape before reconstruction (Lomenie, 2001), as illustrated in Fig. 9. Today's systems of rapid prototyping are technologies used to build physical objects directly from CAD data sources. They are heavily used by design engineers to make rapid tools to manufacture their products in aerospace or motor car industries for instance. In the distant future, biologists, architects, artists but also individuals should be able to automatically manufacture objects of every description with no limit of complexity or input data. For example, by means of today's 3D printers, anyone should be able to copy a 3D object captured by a special stereoscopic device for personal use. Of course this scenario is a long way off. Commercial packages have a lot of limitations (NMC, 2004), and they are still way from making the capture of real-world objects into 3D a trivial task, but we think that the presented mesh filtering operators can help to model intricate organic shapes or regular polygonal objects placed in a cluttered environment.

5.2. Notes on implementation

The complexity of all these operators is the same as that of the Delaunay triangulation : at worst in $O(N \log(N))$ (Boissonnat & Yvinec, 1995) for a set of N points. Besides, determining the triangle and vertex adjacency relations is crucial. For this reason the data structure we use is the quadedge data structure introduced by Guibas and Stolfi

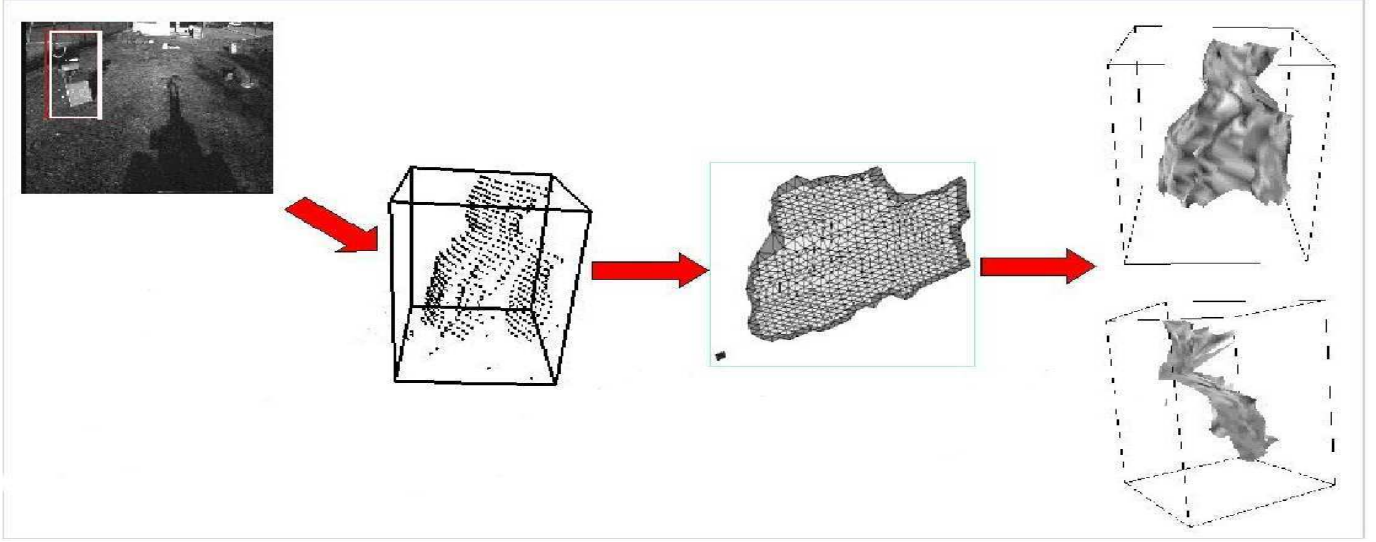


Fig. 9. 3D reconstruction process of an obstacle (the chair) using the designed mesh analysis operators. From left to right : Original image with a zoom on a chair-shaped cluster/obstacle - Stereo-reconstruction of the cluster associated to the obstacle - Projection of the 3D point set onto its eigen plane and morphological mesh opening - Two views of the 3D retroprojection of the opened mesh

(Guibas & Stolfi, 1985) where edges play a leading role. It allows an edge-to-edge navigation through the mesh by means of its algebraic operations. The quadedge data structure captures all the topological information of the subdivision of a surface; each complete quadedge is composed of four branches which are connected together in anticlockwise order by means of the Rot operator. Last, to create and modify the graph only two basic functions are used : "MakeEdge" and "Splice" to connect or disconnect edges.

This data structure enables to consider an extension of edge algebra with basic operations on graphs. We consider it is a good framework to easily extend this algebra to a wide graph toolbox including morphological operations. More specifically, as suggested by (Shewchuk, 1996), we used a recasting of the quadedge data structure dedicated to triangular meshes : the tri-edge structure that leads to a framework to handle triangular meshes. This structure is simple, uniform, effective and compact. But, building a triangular mesh which preserves a rapid and valid access to adjacent edges or triangles using the low-level construction operators is neither simple nor intuitive.

In particular, building Delaunay triangulations is a classic computational geometry problem. Both managing adjacency problems and controlling computational complexity requires a lot of expertise. For instance, worst case time complexities are generally given, but such analyses, from the point of view of the application programmer, are not always sufficient to make the correct decisions. In fact, theoretically better algorithms can sometimes be outperformed by more naive methods; the theoretical asymptotic worst case complexity sometimes fails to consider the optimization techniques that can be applied to reduce the expected complexity. Delaunay triangulation algorithms can be classified as follows :

- higher dimensional embedding (Avis & Bremner, 1995);

- *on-line* incremental insertion (Edelsbrunner & Shah, 1992), holding the theoretical lower worst case time complexity and being simple to program;
- incremental construction (Dobkin & Laszlo, 1989);
- divide and conquer, for which managing adjacency is a rather hard problem during the merging phase. But, for instance, the DeWall algorithm (Cignoni *et al.*, 1998) does not guarantee worst case optimality although it offers good performances in practical situations with optimization techniques.

Nevertheless, the DeWall algorithm fails in easily managing the adjacency tri-edge structures in some specific but common merging phases at the *splice* function level (see Fig. 10). When splicing the newly created triangle (10,11,15) to the rest of the mesh at the level of the vertex 15, the DeWall Algorithms fails in maintaining the adjacency structure between triangles. As a result, an optimized incremen-

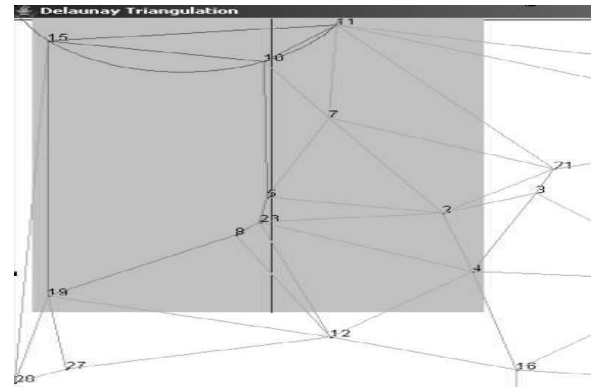


Fig. 10. Specific configuration of the merging phase that leads to an algorithmic problem in managing the adjacency structure with the use of a Divide and Conquer algorithm such as the DeWall one. The problem occurs when splicing for the newly created triangle (10,11,15) at the level of the vertex 15

tal construction algorithm associated with the tri-edge data structure has been preferred in our experiments ².

6. Conclusions and future research

In this paper, we propose the basis of a broad framework for extending the classic manipulations on images like binarization or morphological transformations to surface-like point sets and triangular meshes. Very little work has been dedicated to this issue and always in a non tractable way from an algorithmic point of view. The purpose of this work is also to design a realistic and pragmatic framework for new projects dealing with meshes that users need to structurally clean out or transform (a Java applet and its source code are available in the public domain). Remarkably, the designed algorithms perform as well on the edges as on the triangles of the mesh.

Nevertheless, some theoretical and practical perspectives remain. Unlike classical mathematical morphology, regular lattice is not available and points are not organized, so that the notion of neighborhood system is not easy to manipulate. There is no more 4-connexity or 8-connexity, and that forced us to consider here the neighborhood defined by the implicit structure of Voronoï diagrams, that is the triangles of *Del* as structuring elements. This prevents from easily adapting the shape of structuring elements, which is an essential point in classical morphology theory. We would like to go further in the definition of neighborhood in meshes from a theoretical point of view. And, lastly, we aim to make mesh manipulation as easy as image manipulation : for example, how to easily perform a chain of morphological operators on meshes in a toolkit platform dedicated to meshes whose name could be MeshJ in reference to the famous ImageJ platform.

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² To test the presented results, a Java applet and its source code are available in the public domain at <http://sip-crip5.org/10mn/> in order to be used as the basis for many new projects.

- ulation of general subdivisions and the computation of vorono diagrams. *ACM Transactions on Graphics*, **4**(2), 74–123.
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